10sor network workshop, 17 May, 2015 @ Tokyo Univ. Kashiwa

Geometric Structure of MERA networks: Relation to AdS/CFT Correspondence

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Classification of Tensor Network States

Tensor product states (TPS), Tensor network states (TNS)
→ Variational ansatz satisfying the entanglement entropy scaling

Gapped quantum systems
→ PEPS class
PEPS: Projected Entanglement Pair States
Entanglement entropy: area law scaling $S \propto L^{d-1}$

Quantum critical systems
→ MERA class
MERA: Multi-scale Entanglement Renormalization Ansatz
Entanglement entropy: logarithmic correction (d=1)
Holzhey-Larsen-Wilczek, Calabrese-Cardy

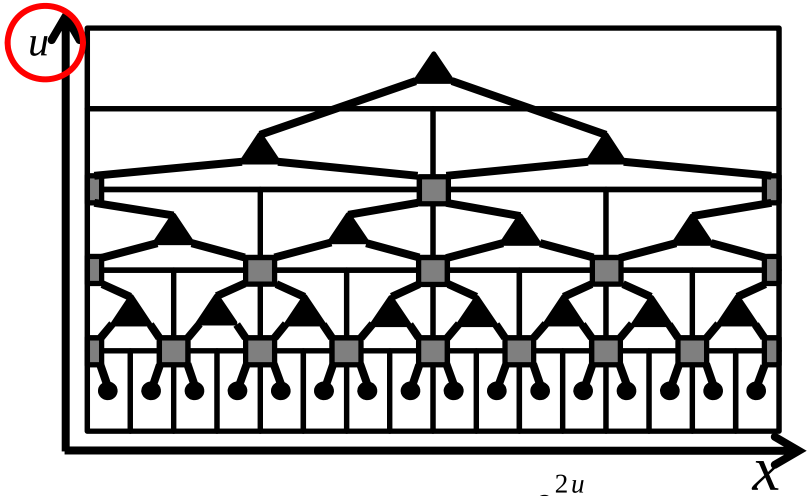
$$S = \frac{c}{3} \log L$$

Purpose of This Talk

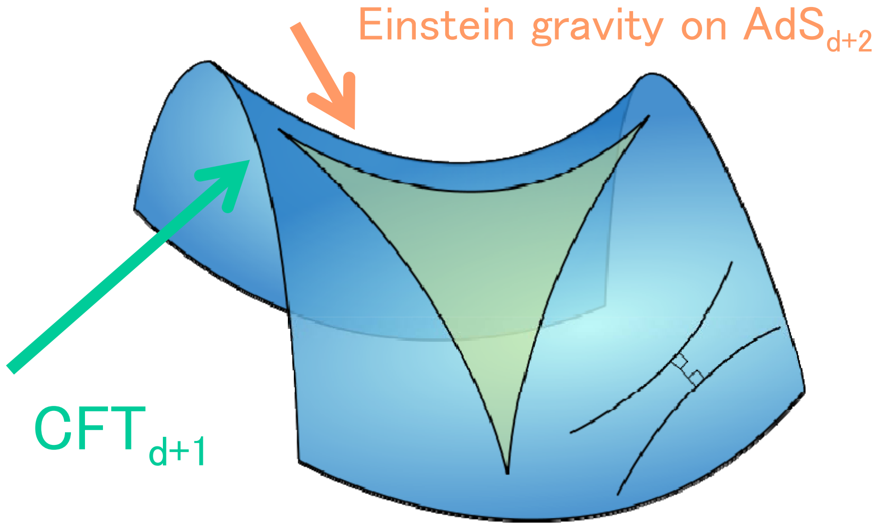
MERA ⟺ **AdS/CFT**

variational ansatz applicable to quantum critical systems

quantum-classical correspondence in string theory



$$ds^2 = g_{uu}\,du^2 + \frac{e^{2u}}{\varepsilon^2}dx^2$$



$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + \eta_{ij}\,dx^i dx^j\right)$$

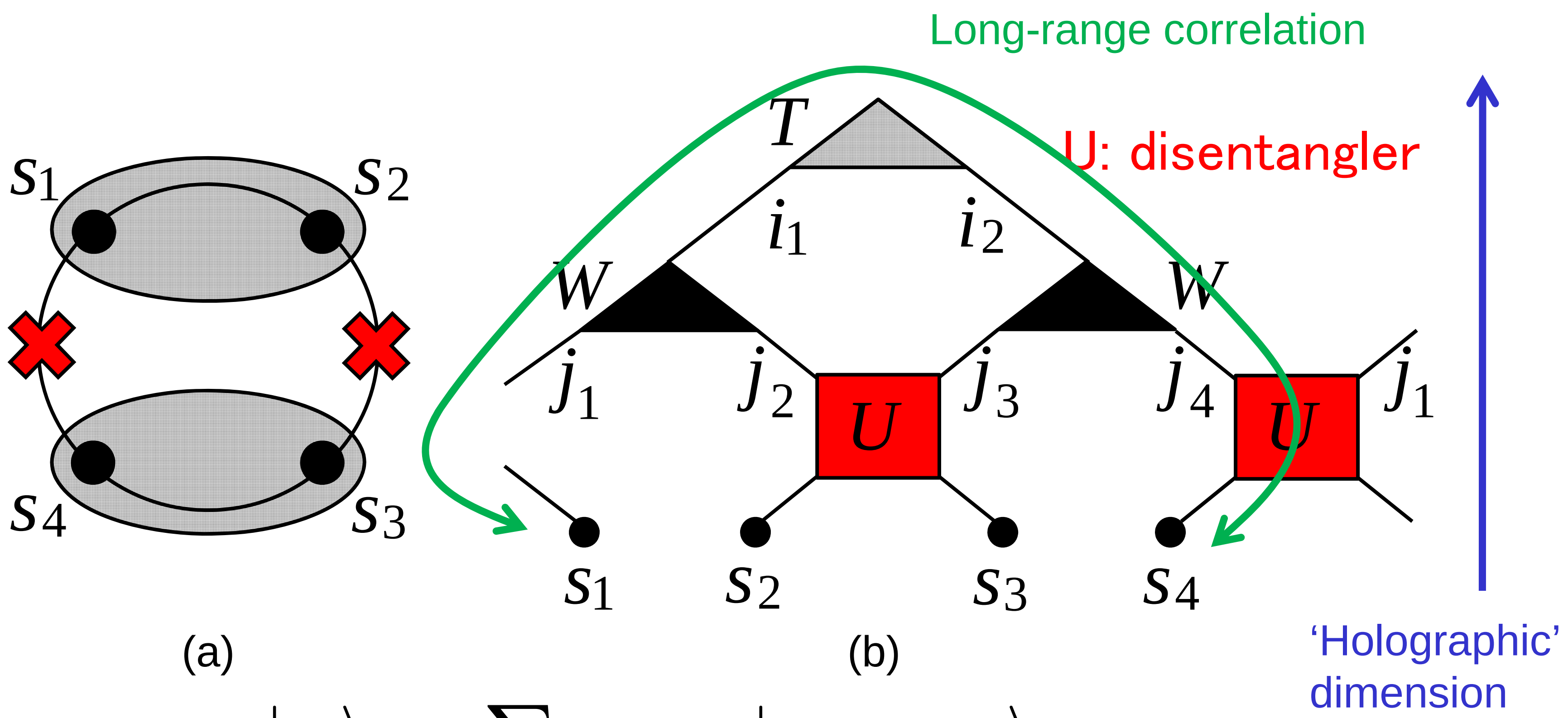
Question: how to examine their similarity ?

(a) Information geometry for MERA network (Takayanagi)

(b) Thermo-field double of MERA ⇔ BTZ black hole (HM)

Hierarchical Tensor Network

Multiscale Entanglement Renormalization Ansatz (MERA)

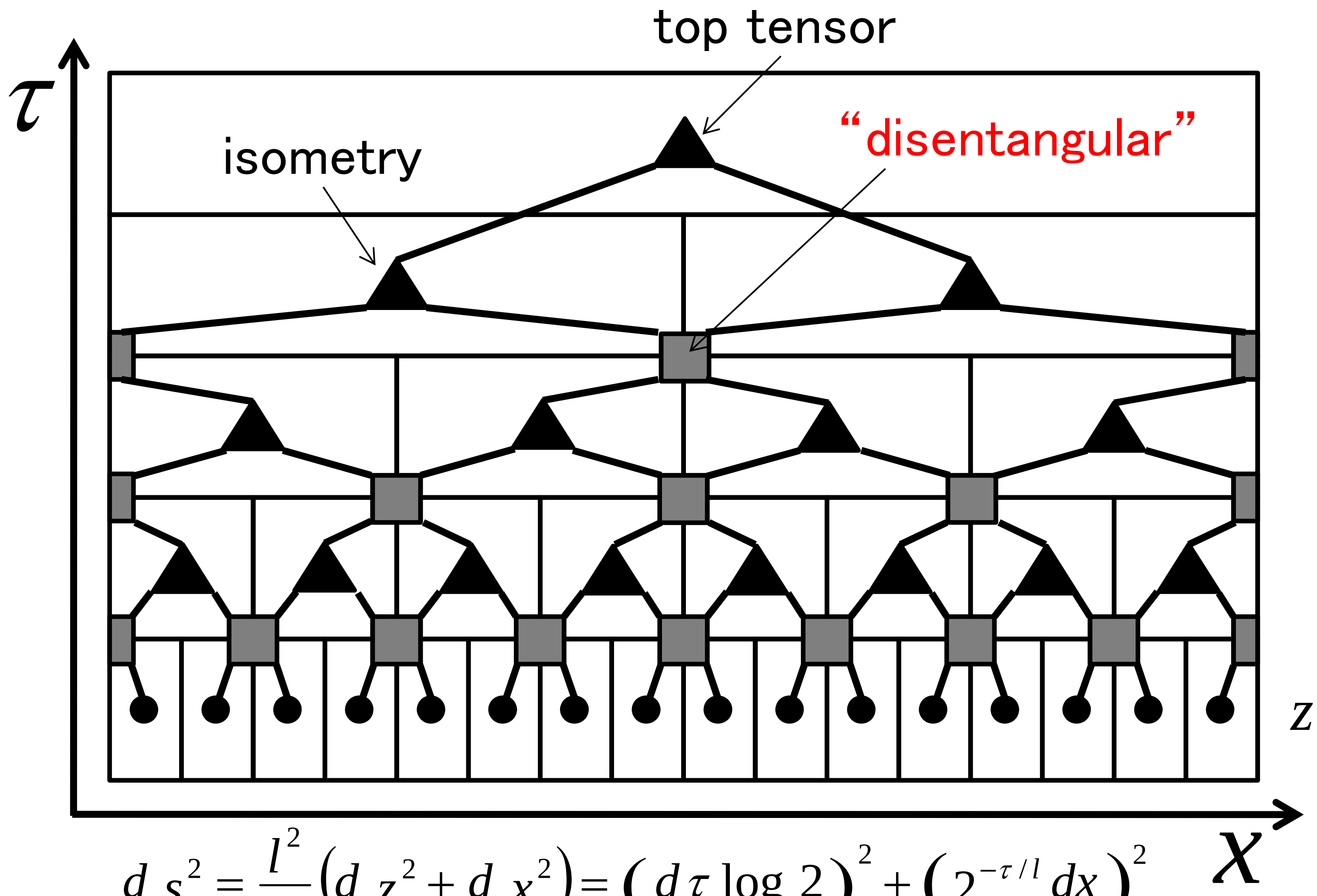


(a) (b)

$$|\psi\rangle = \sum_{s_1,s_2,s_3,s_4,} T_{s_1 s_2 s_3 s_4} |s_1\, s_2\, s_3\, s_4\rangle$$

$$|\psi\rangle = \sum_{i_1,i_2}\ \sum_{j_1,\dots,j_4}\ \sum_{s_1,\dots,s_4} T_{i_1 i_2} W^{i_1}_{j_1 j_2} W^{i_2}_{j_3 j_4} U^{j_2 j_3}_{s_2 s_3} U^{j_4 j_1}_{s_4 s_1} |s_1\, s_2\, s_3\, s_4\rangle$$

(Multiscale Entanglement Renormalization Ansatz, MERA)



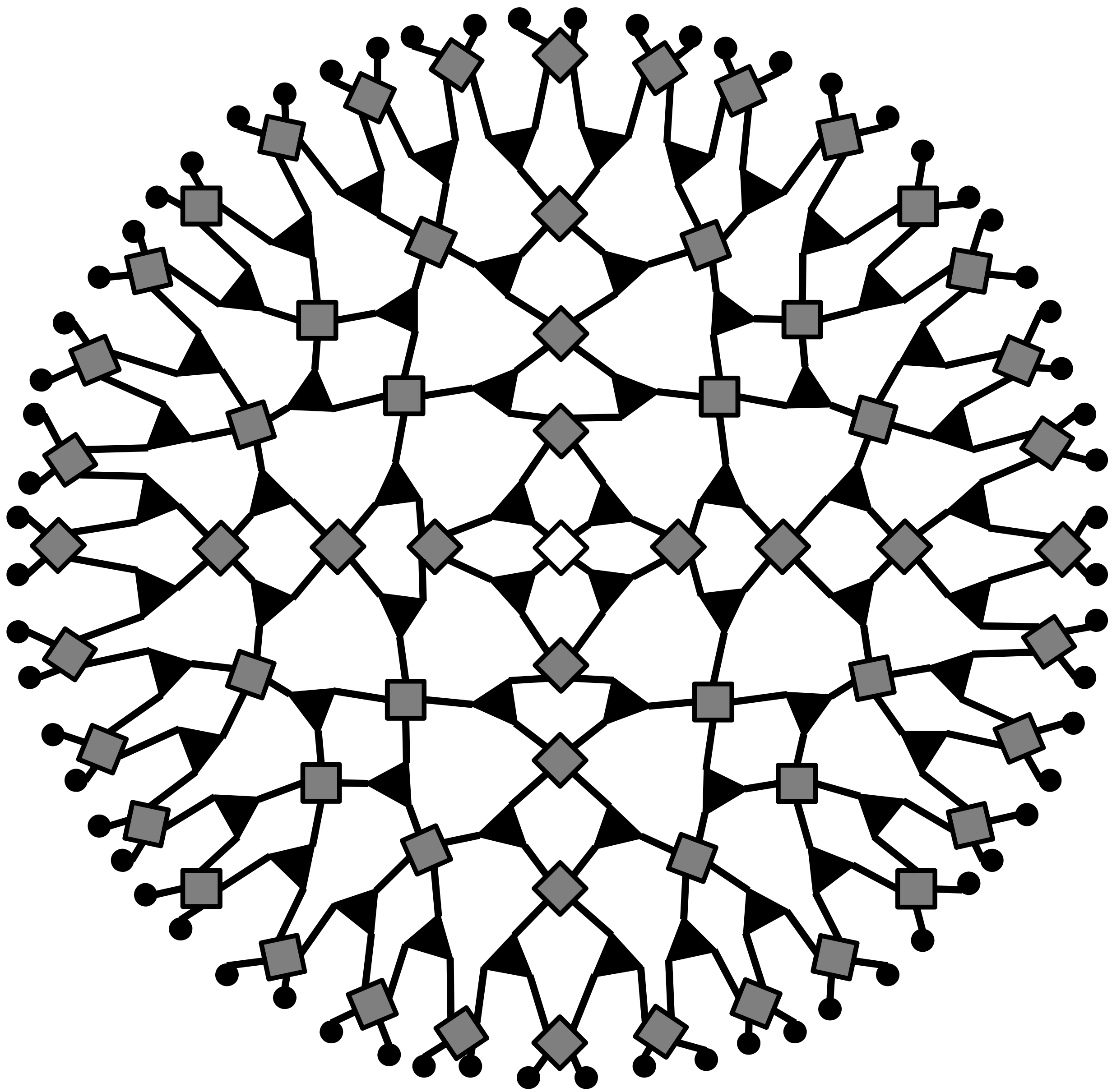
$$z = 2^{\tau / l}\, l$$

$$d s^2 = \frac{l^2}{z^2}\left(d z^2 + d x^2\right) = \left(d \tau \log 2\right)^2 + \left(2^{-\tau / l} dx\right)^2$$

MPS → decomposed into many tensors with different function

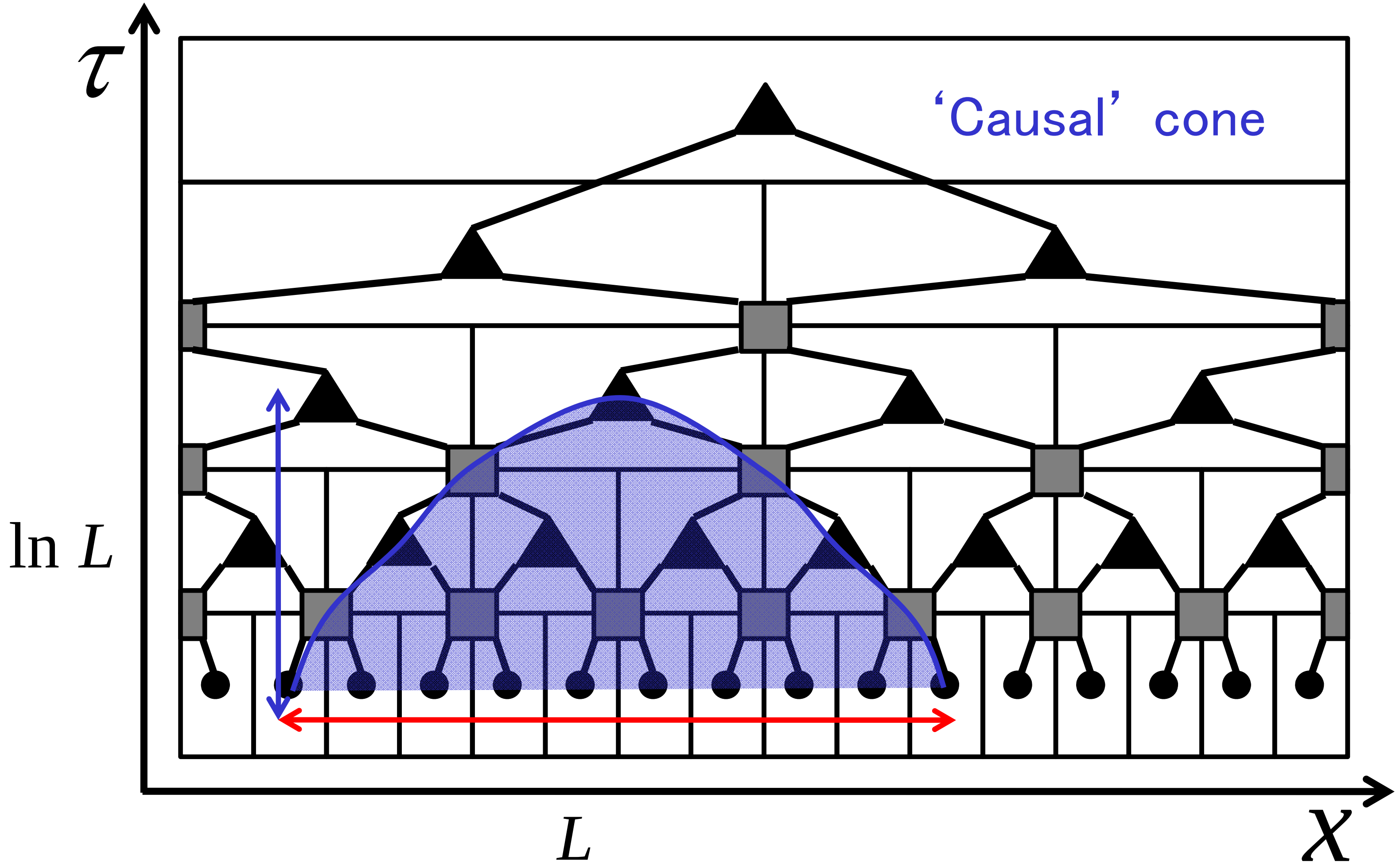
Basis change (disentangler) before renormalization

Poincare Disk Model for MERA Network



How to evaluate entanglement entropy in holographic space ?

Close connection to 'Ryu–Takayanagi formula' developed in superstring theory



S = minimal surface area in holographic space

Binary decomposition

↓

Spatially 1D cases: $$\underbrace{2 + 2 + \cdots + 2}_{\text{No. of boundary points: } \ln L} = 2 \ln L$$

Spatially 2D cases:

$$4L\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right) = 4L\left(2 - \frac{1}{2^n}\right) \to 8L$$

Anti-de Sitter (Hyperbolic) Space and CFT

$$\eta_{ij} = \begin{pmatrix} -1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Metric of AdS space   z: radial axis, z→0: boundary

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \frac{l^2}{z^2}\left(dz^2 + \eta_{ij}\, dx^i dx^j\right)$$

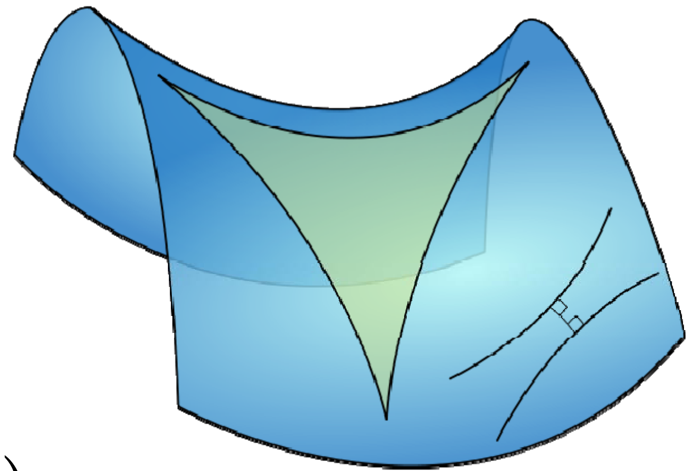
$$ds \sim \frac{l}{z} dz$$

$$s \sim l \log z$$

Infinitesimal trans.

$$\bar{x}^i = x^i + \xi^i(x)$$

$$\bar{z} = z + z\zeta(x)$$



$$z \to 0$$

$$d\,\bar{s}^2 = \bar{g}_{\mu\nu}\, d\,\bar{x}^\mu d\,\bar{x}^\nu$$

$$= d\,s^2 + \left(\partial_i \xi_j + \partial_j \xi_i - 2\zeta\, \eta_{ij}\right) d\,x^i d\,x^j$$

Isometry trans. ⇒ conformal Killing equation at z→0

Boundary of AdS_{d+1} ⇒ CFT_d

AdS/CFT correspondence and Holographic Entropy

Gubser–Klevanov–Polyakov(GKP)–Witten relation

$$\langle O(x_1)\cdots O(x_n)\rangle_{CFT} = \frac{\delta}{\delta\phi(x_1)}\cdots\frac{\delta}{\delta\phi(x_n)}\exp\left(-\frac{1}{2\kappa}I(\phi(x))\right)\Big|_{\phi=0}$$

Ryu–Takayanagi Formula (extension of Beckenstein–Hawking)

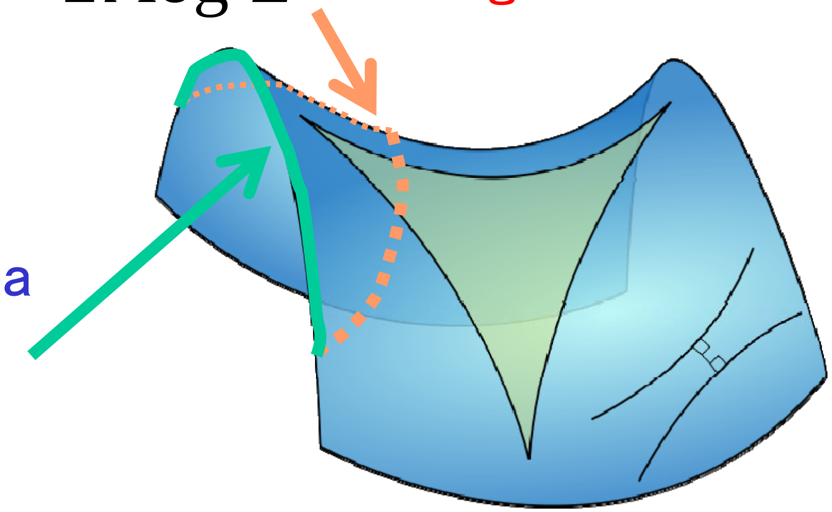
2006 $$S = \frac{\gamma}{4G}$$

$$\gamma = 2l\log L$$ 2D: geodesic distance

γ : area of minimal surface

Calabrese–Cardy formula

$$S = \frac{1}{3}c\log L$$



$$c = \frac{3l}{2G}$$ Brown–Henneaux central charge

Geometry of quantum states

Difference between two similar states
→ distance on information space (Bures metric)

$$D(\theta) = 1 - \left|\left\langle \psi(\theta) \middle| \psi(\theta + d\theta) \right\rangle\right|^2$$

Θ: vector valued, internal parameters of Ψ

Expansion of D by dθ up to the second order

$$D(\theta) = \chi_{\mu\nu}(\theta) d\theta^{\mu} d\theta^{\nu}$$

$$\chi_{\mu\nu}(\theta) = \left\langle \partial_\mu \psi(\theta) \middle| \partial_\nu \psi(\theta) \right\rangle - \left\langle \partial_\mu \psi(\theta) \middle| \psi(\theta) \right\rangle \left\langle \psi(\theta) \middle| \partial_\nu \psi(\theta) \right\rangle$$

$$\chi_{\mu\nu}(\theta) = \chi^{*}_{\nu\mu}(\theta)$$

Fisher metric g and Berry curvature F

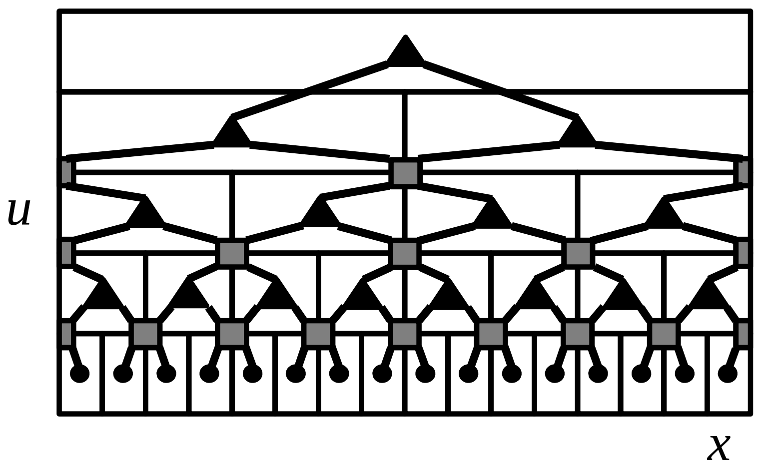
$$g_{\mu\nu}(\theta) = \frac{\chi_{\mu\nu}(\theta) + \chi_{\nu\mu}(\theta)}{2} \qquad F_{\mu\nu}(\theta) = i\left(\chi_{\mu\nu}(\theta) - \chi_{\nu\mu}(\theta)\right)$$

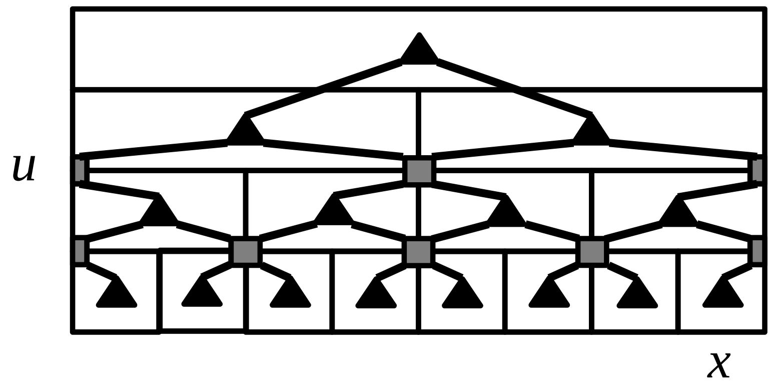
Emergent AdS spacetime from continuous MERA

Question:

Does the holographic direction of MERA network really correspond to the radial axis of AdS spacetime ?

$u \subset \theta$





→ Calculation of $g_{uu}(u)$

$$D(u) = 1 - \left| \langle MERA(u) | MERA(u+du) \rangle \right|^2$$

RG changes the number of effective lattice sites and it is hard to take the inner product directly → field-theoretical treatment

Physical Information and Geometric Distance

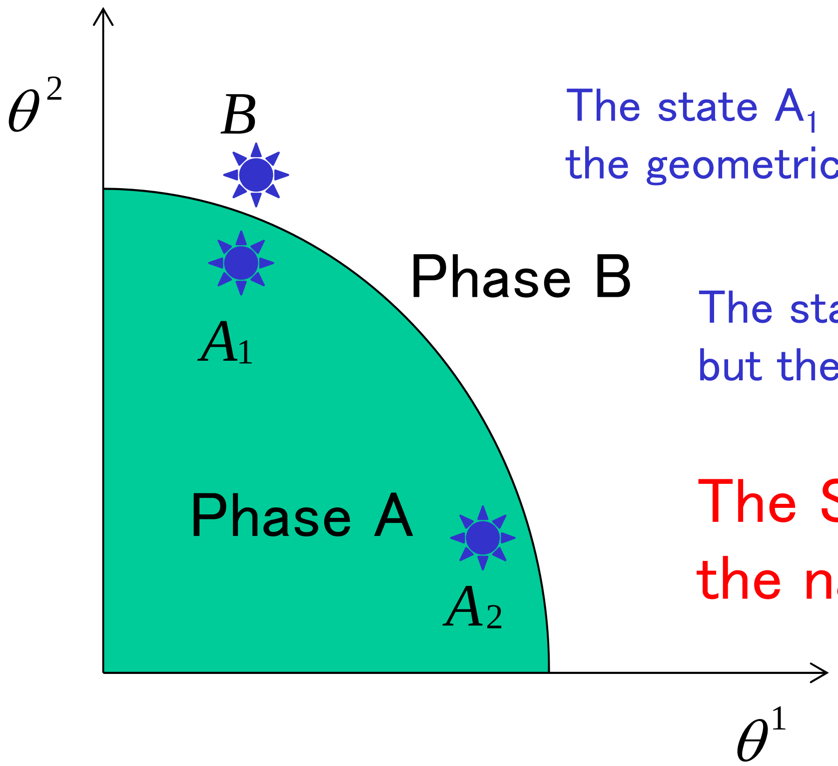
Phase diagram for states of matter

→ 'highly compressed' quantum information

Only the information entropy (amount of information, not quality) has some physical meaning.

Equation of states for ideal gas $PV = nRT$

(Avogadro number → only three parameters P,V,T)



The state A_1 is different from the state B, but the geometric distance on this phase diagram is short.

The states A_1 and A_2 are in the same phase, but their distance is far.

The Shannon entropy can identify the nature of each phase.

Relative information entropy → Fisher Metric

Probability distribution $\sum_n p_n(\theta) = 1$

θ : internal parameter (vector valued)

→ This parameter set determines a particular physical state.

'Relative' information entropy

$$D(\theta) = -\sum_n p_n(\theta) \log p_n(\theta) + \sum_n p_n(\theta) \log p_n(\theta + d\theta)$$

$$\approx \frac{1}{2} \sum_n p_n(\theta) \frac{\partial \log p_n(\theta)}{\partial \theta^\mu} \frac{\partial \log p_n(\theta)}{\partial \theta^\nu} d\theta^\mu d\theta^\nu$$

$$\gamma_n(\theta) = -\log p_n(\theta)$$

Fisher metric

$$g_{\mu\nu}(\theta) = \sum_n p_n(\theta) \frac{\partial \log p_n(\theta)}{\partial \theta^\mu} \frac{\partial \log p_n(\theta)}{\partial \theta^\nu} = \langle \partial_\mu \gamma \, \partial_\nu \gamma \rangle$$

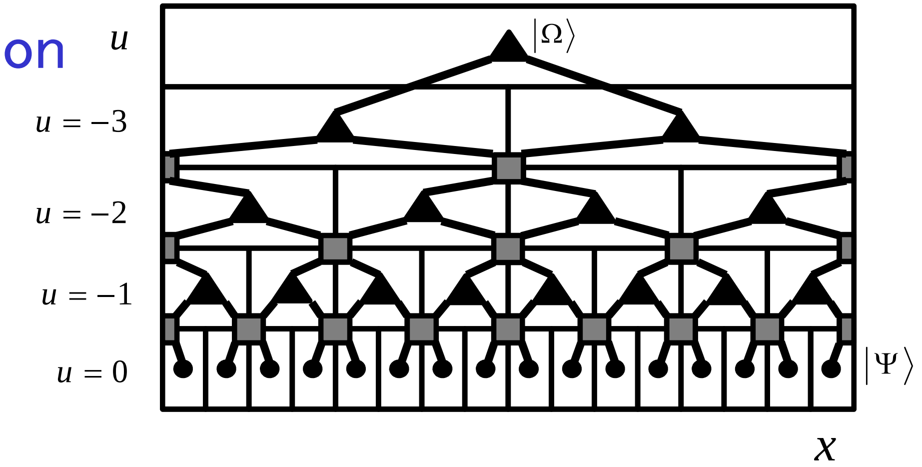
cMERA

Continuous version of MERA (cMERA)

Evolution operator along holographic direction

$$U(u_1, u_2) = P \exp\left[-i\int_{u_1}^{u_2}\left(K(u) + L\right)du\right]$$

K: entangler, L: scale transformation



$$k \leq \Lambda e^{u}$$

IR & UV limit

$$|\Omega\rangle = |\Psi(u_{IR})\rangle, \; |\Psi\rangle = |\Psi(u_{UV} = 0)\rangle$$

Any state with a particular scale u is constructed by the evolution from a reference state at IR or UV.

$$|\Psi\rangle = U(0,u)|\Psi(u)\rangle \qquad |\Psi(u)\rangle = U(u,u_{IR})|\Omega\rangle \qquad L|\Omega\rangle = 0$$

Interaction representation

$$K_I(u) = e^{iuL} K(u) e^{-iuL} \qquad U(u_1, u_2) = e^{-iu_1 L} P \exp\left[-i\int_{u_1}^{u_2} K_I(s) ds\right] e^{iu_2 L}$$

$$|\Psi(u)\rangle = e^{-iuL} P \exp\left[-i\int_{u_{IR}}^{u} K_I(s) ds\right] |\Omega\rangle = e^{-iuL} |\Phi(u)\rangle$$

<u>Explicit form of entangler (assumption) for 1D free scalar case</u>

$$H = \frac{1}{2}\int dk \left[\pi(k)\pi(-k) + \varepsilon_k^2 \phi(k)\phi(-k)\right] \qquad \varepsilon_k = \sqrt{k^2 + m^2}$$

$$K(u) = \frac{1}{2}\int dk g(k,u)\left[\phi(k)\pi(-k) + \pi(k)\phi(-k)\right]$$

Energy minimization

Scale invariance of Ω

Scaling properties of Φ, π

$$\Downarrow$$

$$g(k,u) \sim \chi(u)\theta(\Lambda - |k|)$$

$$k \leq \Lambda e^{u}$$

$$\chi(s) = \frac{1}{2}\frac{e^{2u}}{e^{2u} + m^2/\Lambda^2}$$

Derivation of Bures metric

$$g_{uu}(u) = \langle \Phi(u) | K_I(u)^2 | \Phi(u) \rangle - \langle \Phi(u) | K_I(u) | \Phi(u) \rangle^2$$

$$g_{uu}(u) = \chi(u)^2 = \frac{e^{4u}}{4\left(e^{2u} + m^2/\Lambda^2\right)^2}$$

$$e^{2u} = \frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2}$$

$$ds^2 = \frac{dz^2}{4z^2} + \left(\frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2}\right) dx^2 + g_{tt}\, dt^2$$

Massless case → pure AdS

Finite mass → truncation of the network $\quad z < \frac{1}{m}$

Thermofield dynamics (TFD) for finite-T wavefunction

Purpose: finite-T MERA and its relation with AdS/CFT

Finite-T → thermal avarage

TFD form → 'thermal vacuum'

Identity state (maximally entangled) $\left| I \right\rangle = \sum_n \left| n \right\rangle \otimes \left| \tilde{n} \right\rangle$

General representation theorem

$$\left| I \right\rangle = \sum_n \left| n \right\rangle \otimes \left| \tilde{n} \right\rangle = \sum_\alpha \left| \alpha \right\rangle \otimes \left| \tilde{\alpha} \right\rangle$$

$$\left| O(\beta) \right\rangle = \rho^{1/2} \left| I \right\rangle$$

$$\left\langle O(\beta) \right| A \left| O(\beta) \right\rangle = \sum_{m,n} \left\langle m\tilde{m} \right| \rho^{1/2} A \rho^{1/2} \left| n\tilde{n} \right\rangle$$

$$= \sum_{m,n} \left\langle m \right| \rho^{1/2} A \rho^{1/2} \left| n \right\rangle \delta_{\tilde{m}\tilde{n}} = tr\left(\rho A\right)$$

Thermal state in TFD

$$\left|\psi(\beta)\right\rangle = \sum_{\{m_j\}} \sum_{\{\tilde{n}_j\}} c^{\{m_j\}\{\tilde{n}_j\}} \left| \{m_j\}\{\tilde{n}_j\} \right\rangle$$

Singular value decomposition

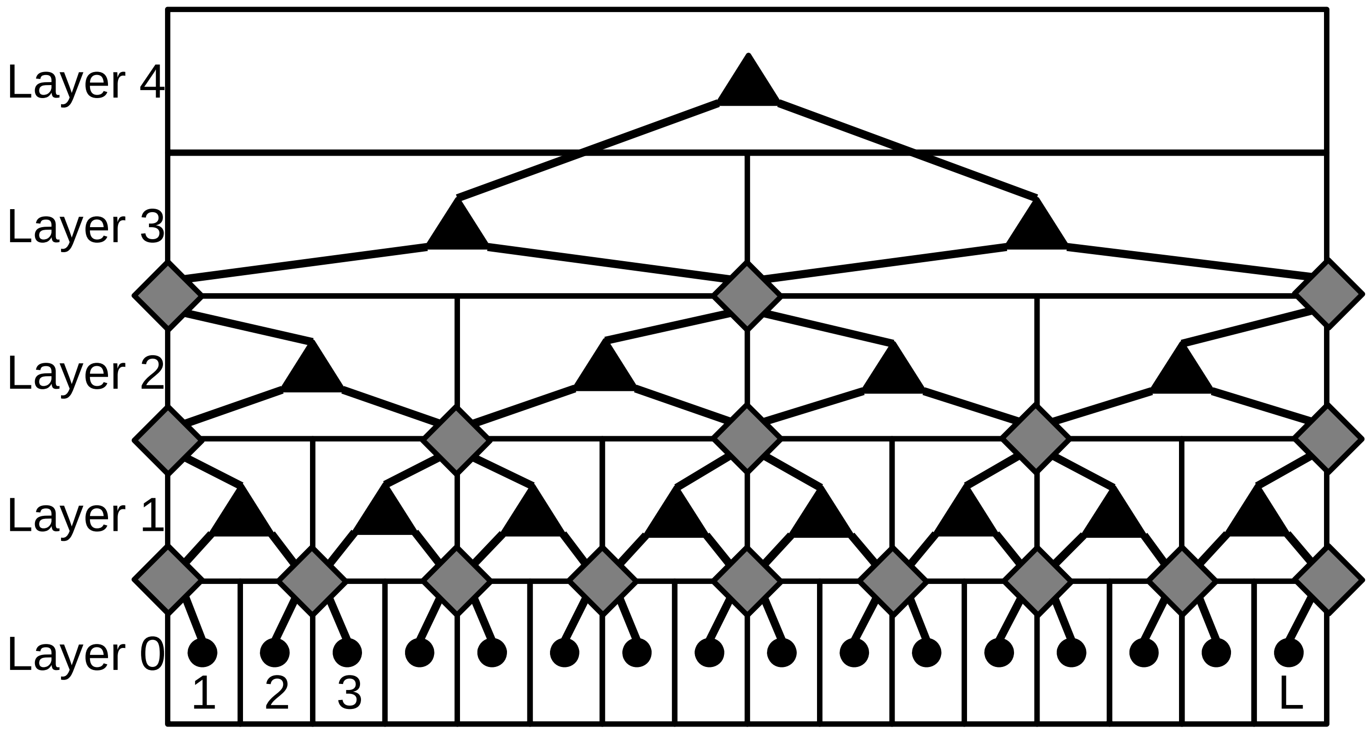
$$c^{\{m_j\}\{\tilde{n}_j\}} = \sum_{\alpha=1}^{\chi} A_{\alpha}^{\{m_j\}} A_{\alpha}^{\{\tilde{n}_j\}}$$

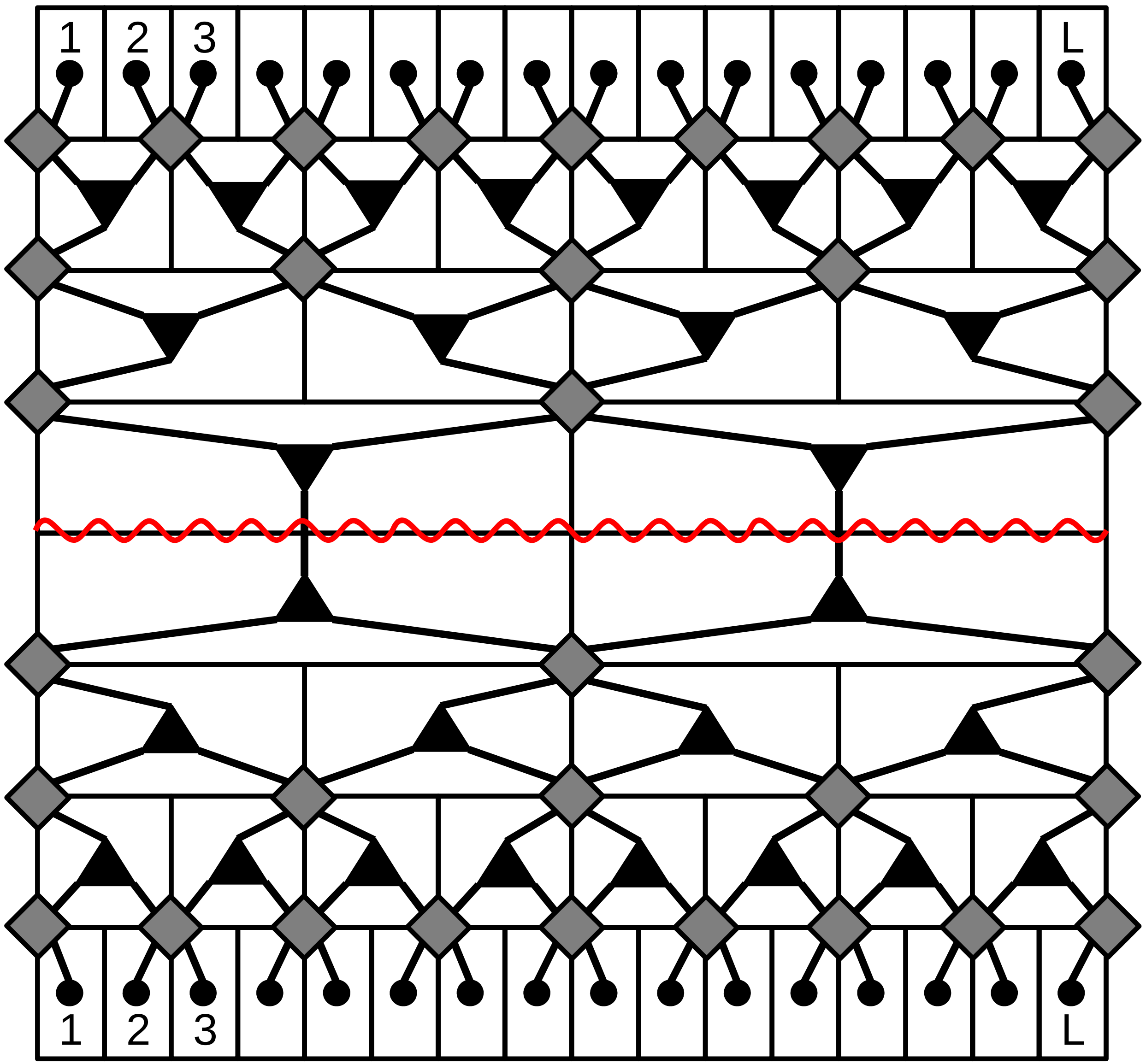
α : event horizon → black hole entropy

= maximally entanglement entropy

Imagine Penrose diagrams …

T=0

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Banados-Teitelboim-Zanelli (BTZ) metric: black hole solution

Exact solution of Einstein eq. in (2+1)D
with negative cosmological term

* Schwarzschild solution
  in (3+1)D flat spacetime

$$f(z) = 1 - \frac{a}{z}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0$$

$$ds^2 = \frac{1}{z^2}\left(-f(z)\,dt^2 + \frac{1}{f(z)} dz^2 + dx^2 \right)$$

Event horizon: $z = z_H$

$$f(z) = 1 - \left(\frac{z}{z_H} \right)^2$$

f(z)=1 → anti-de Sitter (AdS) spacetime

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx^2 \right)$$

Maximally-extension of BTZ spacetime

Coordinate transformation

$$ds^2 = \frac{1}{z^2}\left(-f(z)\,dt^2 + \frac{1}{f(z)}dz^2 + dx^2 \right)$$

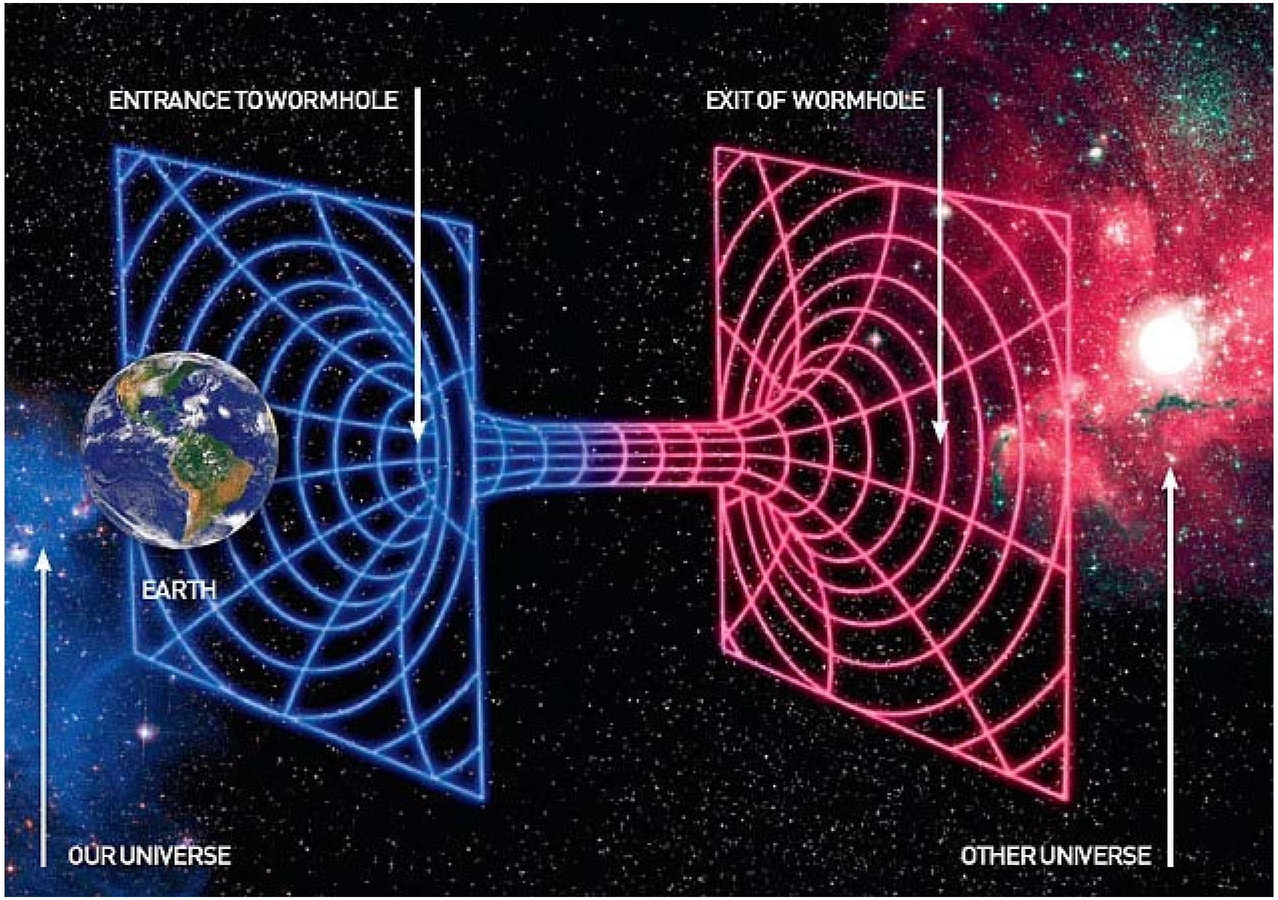
$$j = \ln\left(\frac{2z/\varepsilon}{1+\sqrt{f(z)}} \right) \qquad j_H = \ln\left(\frac{2z_H}{\varepsilon} \right)$$

ε : UV cut-off

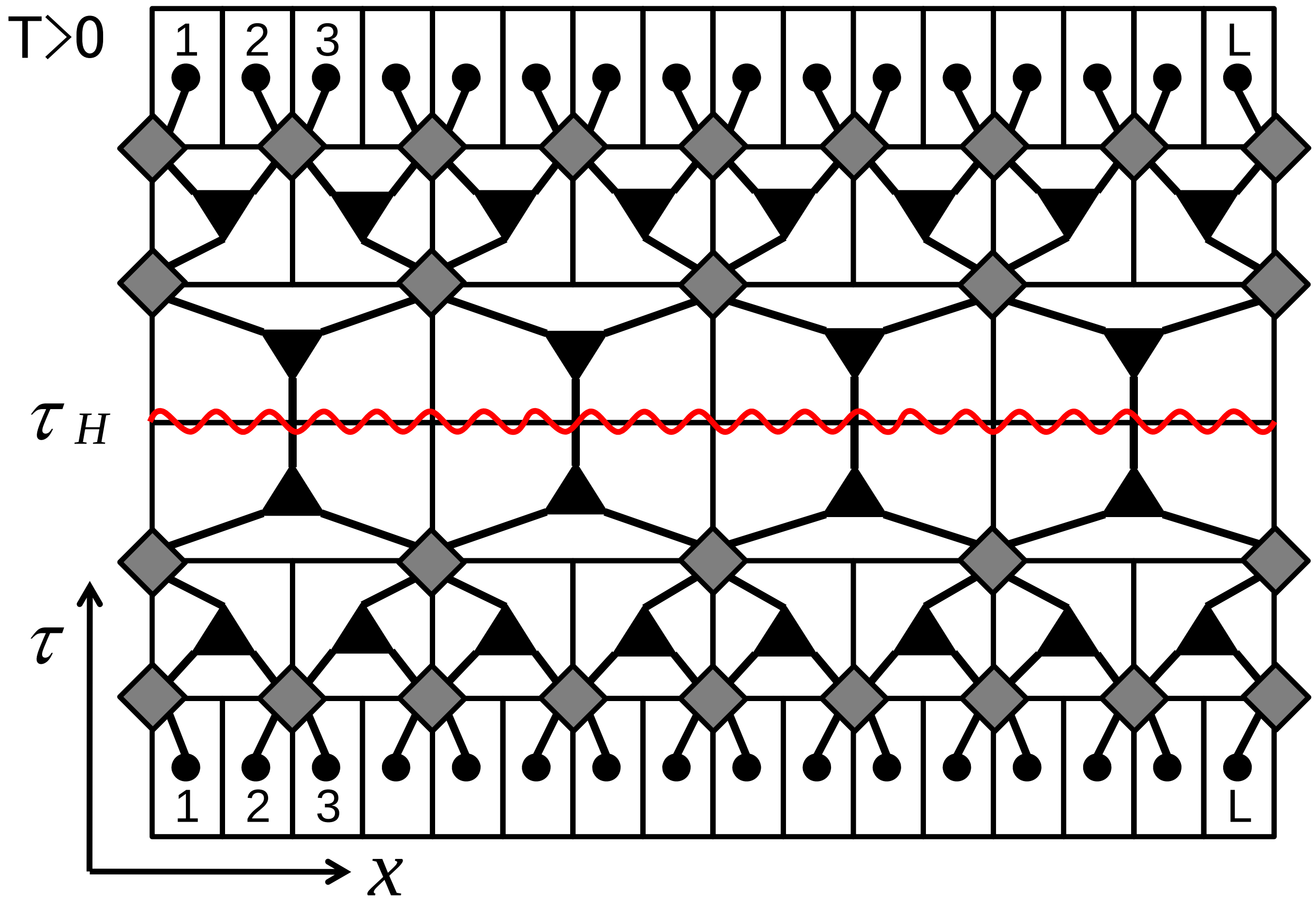
$$ds^2 = -h(j)\,dt^2 + d(j\ln\eta)^2 + g(j)\,dx^2$$

$$\alpha = j - j_H \qquad \beta = 2e^{-j_H}\frac{x}{\varepsilon}$$

$$ds^2 = d\alpha^2 + (\cosh\alpha)^2 d\beta^2$$


ENTRANCE TO WORMHOLE
EXIT OF WORMHOLE
EARTH
OUR UNIVERSE
OTHER UNIVERSE

Finite-T MERA Network and AdS Black Hole



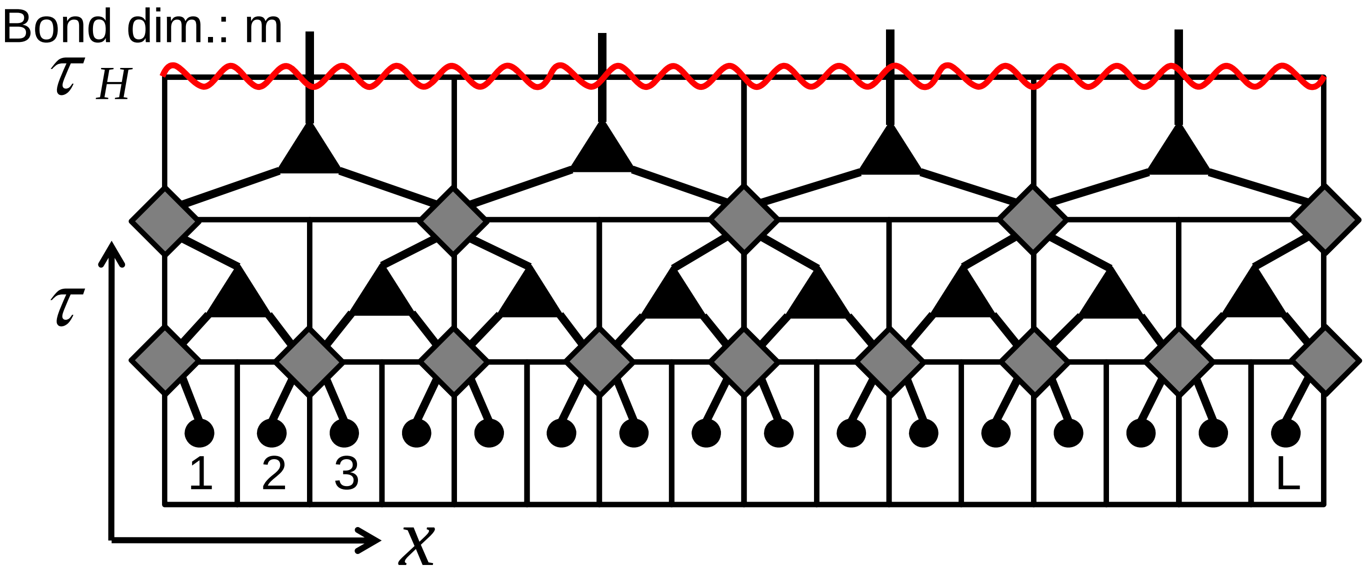
Vertical axis = energy scale, temperature scale

Wave function approach at finite-T → thermofield dynamics

→ Connection between original and tilde spaces

Temperature of MERA Network

Truncation of upper MERA layers = AdS black hole



$z = 2^{\tau}$

Area of interface: $\frac{L}{2^{\tau_H}} = A$ Total dim. at interface: $\chi = m^A$

Beckenstein–Hawking entropy & Calabrese–Cardy formula:

$$S_{BH} = A \ln m = \frac{L}{z_H} \ln m$$

$$S_{CFT} = \frac{c}{3} \ln\left(\frac{\beta}{\pi\varepsilon} \sinh\left(\frac{\pi L}{\beta} \right) \right)$$

$$k_B T = \left(\frac{3}{c\pi} \ln m \right) \frac{1}{z_H} \propto \frac{1}{z_H}$$

Summary

Information geometrical interpretation of MERA

・Bures metric of MERA ～ radial axis of AdS

・massive case ～ deformation of AdS

Finite-T MERA network

・ Truncation of the network at the IR region

・ layer number of MERA ～ 1/T

・ consistency with field-theoretical treatment

資料集

Target of this talk: Fisher metric and AdS/CFT

Information geometry → Fisher metric

$$g_{\mu\nu}(\theta) = \sum_n p_n(\theta) \frac{\partial \ln p_n(\theta)}{\partial \theta^\mu} \frac{\partial \ln p_n(\theta)}{\partial \theta^\nu}$$

$$\sum_n p_n(\theta) = 1$$

θ : internal parameters properly describing the system

(a) The universal metric in information geometry
(at least in a classical level)

(b) Natural source to produce quantum-classical correspondence

(c) Similarity to Lagrangian of free scaler field

(d) We may construct examples in condensed matter physics.

(e) Point (entropy) → Spacetime (metric+connection)

Questions:

What kind of correspondence comes from information geometry ?

Is the new correspondence related to AdS/CFT ?

1D free scalar field

Hamiltonian

$$H = \frac{1}{2}\int dk \left[\pi(k)\pi(-k) + \varepsilon_k^2 \phi(k)\phi(-k)\right] \qquad \varepsilon_k = \sqrt{k^2 + m^2}$$

IR state

$$\left(\sqrt{M}\phi(x) + \frac{i}{\sqrt{M}}\pi(x)\right)\left|\Omega\right\rangle = 0 \qquad M \approx O(\Lambda), \Lambda : \mathrm{UV\ \ cut\text{-}off}$$

IR state → scale invariant

$$e^{-iuL}\phi(x)e^{iuL} = e^{u/2}\phi(e^u x)$$
$$e^{-iuL}\phi(k)e^{iuL} = e^{-u/2}\phi(e^{-u} k)$$
$$e^{-iuL}\pi(x)e^{iuL} = e^{u/2}\pi(e^u x)$$
$$e^{-iuL}\pi(k)e^{iuL} = e^{-u/2}\pi(e^{-u} k)$$

Assumption for functional form of K(u)

$$K(u)=\frac{1}{2}\int dk\, g(k,u)\left[\phi(k)\pi(-k)+\pi(k)\phi(-k)\right]$$

Effect of entangler on scale dimension

$$U(0,u)^{-1}\phi(k)U(0,u)=e^{-f(k,u)}e^{-u/2}\phi\left(e^{-u}k\right)$$
$$U(0,u)^{-1}\pi(k)U(0,u)=e^{f(k,u)}e^{-u/2}\pi\left(e^{-u}k\right)$$

$$K_I(u)=e^{iuL}K(u)e^{-iuL}=\frac{1}{2}\int dk\, g\left(k\,e^{-u},u\right)\left[\phi(k)\pi(-k)+\pi(k)\phi(-k)\right]$$

$$\frac{\partial f(k,u)}{\partial u}=g\left(k\,e^{-u},u\right)$$

$$E=\langle\Psi|H|\Psi\rangle=\langle\Omega|H(u_{IR})|\Omega\rangle=\frac{1}{4}\int dk\left[e^{2f(k,u_{IR})}M+\frac{\varepsilon_k^2}{M}e^{-2f(k,u_{IR})}\right]$$

$$\frac{\delta E}{\delta\chi(u)}=\int_{|k|\le\Lambda e^{u}}dk\left[e^{2f(k,u_{IR})}M-\frac{k^2}{M}e^{-2f(k,u_{IR})}\right]\Rightarrow f(k,u_{IR})=\frac{1}{2}\log\frac{\varepsilon_k}{M}$$

$$f(k,u_{IR})=\int_0^{u_{IR}}g\left(k\,e^{-s},s\right)ds=\int_0^{-\log\Lambda/|k|}\chi(s)ds\Rightarrow\chi(s)=\frac{1}{2}\frac{e^{2u}}{e^{2u}+m^2/\Lambda^2}$$