10sor network workshop
@Kashiwanoha Future Center

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Symmetry protected topological phases in quantum spin systems

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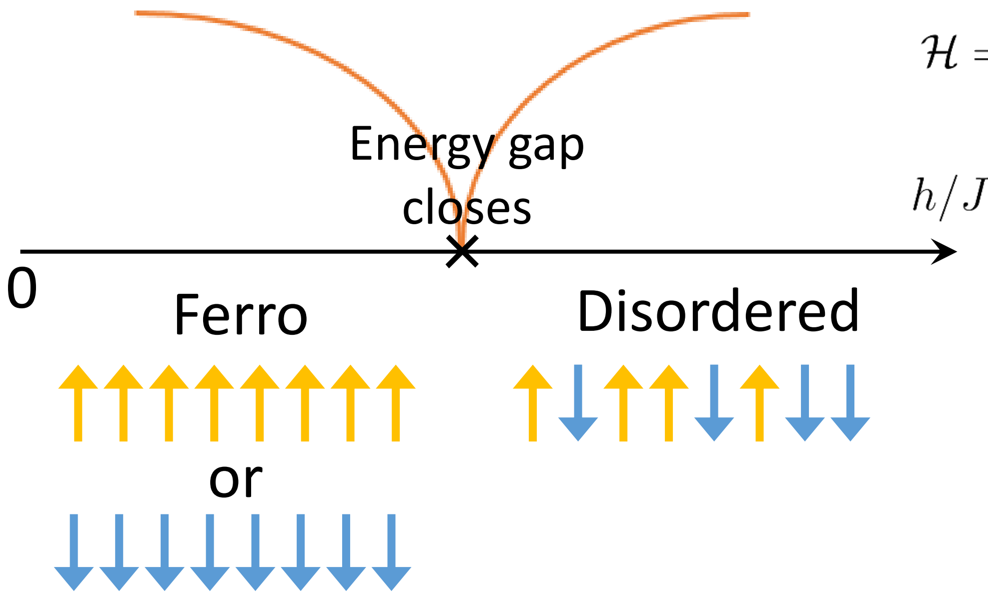
Phase classification

Determination of phase diagrams is an important task in many fields of physics.

What are criteria for the classification of phases?
e.g. Landau theory (symmetry breaking)

$$\mathcal{H} = -J \sum_j S_j^z S_{j+1}^z - h \sum_j S_j^x$$



$$\left\langle \sum_j S_j^z \right\rangle$$

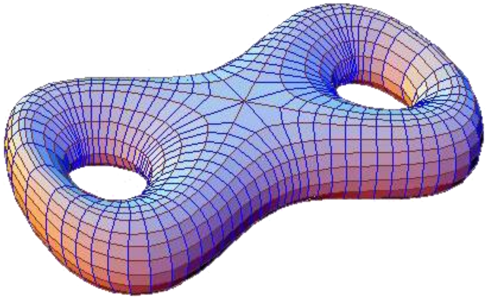
Order parameter
(Magnetization)

(Intrinsic) topological order

A phase which cannot be characterized by (local) order parameters but nontrivial

- Fractionalization of excitation (anyons)
- Ground state degeneracy depending on topology of the manifold on which the system resides

E.g., Fractional quantum Hall (FQH) effect
Gapped quantum spin liquid
(Toric code, Quantum dimer model, etc.)



$\nu=1/3$ FQH state on an orientable surface
3^g-fold degenerate ground states
(g: genus)

Symmetry protected topological phase

- Gapped systems

  Long-range entangled state ・・・ Intrinsic topological order

  Short-range entangled state

  Local unitary transformation ⇓ Without any symmetry

  Trivial (direct product) state

- Short Long range entangled state can be nontrivial if some symmetry is imposed.
  - ➢Local order parameter → Ginzburg-Landau theory
  - ➢No Local order parameter → Symmetry protected topological (SPT) phase

Typical SPT phase — Haldane phase

(1+1) D Heisenberg antiferromagnet (Spin-S)

$$\mathcal{H} = \sum_j J\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \ (J > 0)$$

Effective field theory — O(3) nonlinear sigma model

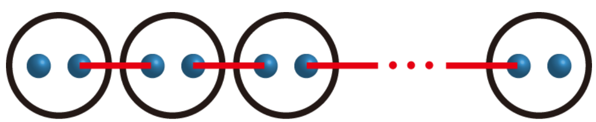
$$S[\boldsymbol{n}] = \frac{1}{2g}\int d\tau dx \left\{\frac{1}{v}(\partial_\tau \boldsymbol{n})^2 + v(\partial_x \boldsymbol{n})^2\right\} + \frac{i\theta}{4\pi}\int d\tau dx \boldsymbol{n}\cdot\partial_\tau\boldsymbol{n}\times\partial_x\boldsymbol{n}$$

$$|\boldsymbol{n}| = 1 \qquad \boldsymbol{S}_j/S \sim (-1)^j \boldsymbol{n}(x) + (a/S)\boldsymbol{l}(x) \qquad g = 2/S \quad v = 2JS$$

Haldane conjecture Haldane, 1983

$\theta = 2\pi S \equiv 0 \mod 2\pi$ Integer spin (gapped)

$\theta \equiv \pi \mod 2\pi$ Half-odd integer spin (gapless, critical)



Q: Are “Gapped phases” all the same? S=1,2,3,...

Typical SPT phase — Haldane phase

A: No

$$\mathcal{H} = \sum_j J \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} \; (J > 0)$$

Integer spin is additionally classified
into odd-integer (SPT) and even-integer (Trivial)

Pollmann-Berg-Turner-Oshikawa, 2010, 2012

S=1 (Haldane phase) ground state is protected by
(A) π-rotation about spin x,y,z-axis ($Z_2 \times Z_2$)
(B) Time-reversal symmetry
(C) Link-center inversion symmetry

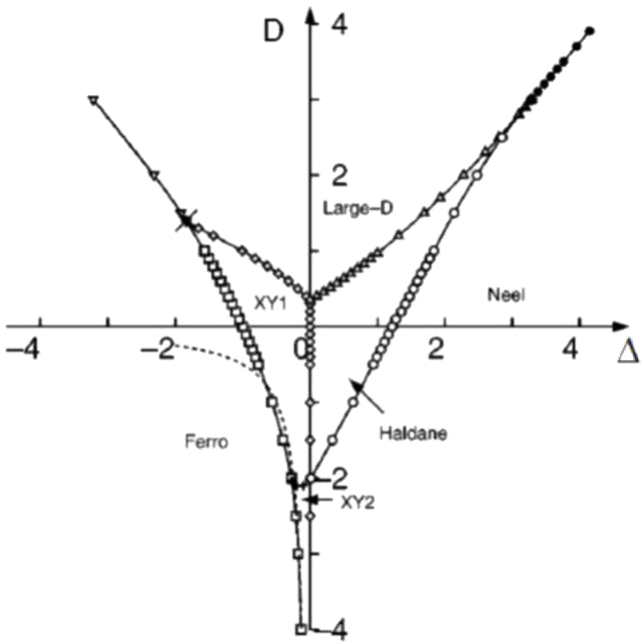
S=2 ground state is smoothly deformed into
trivial (direct product) state.

Phase diagram for S=1 and S=2

$$\mathcal{H} = J\sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) + D\sum_j (S_j^z)^2$$

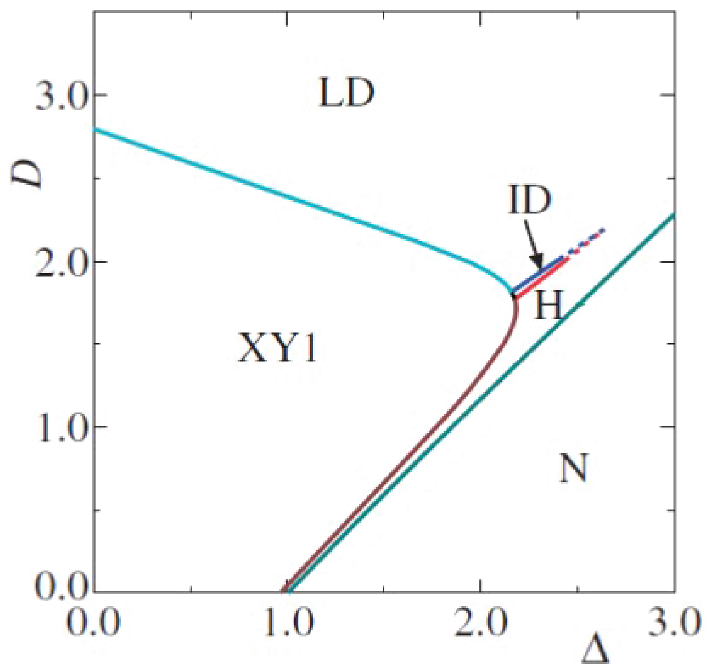
Large-D (trivial) phase $|000\cdots\rangle$ (S^z-basis)

S=1



Chen et al., 2003

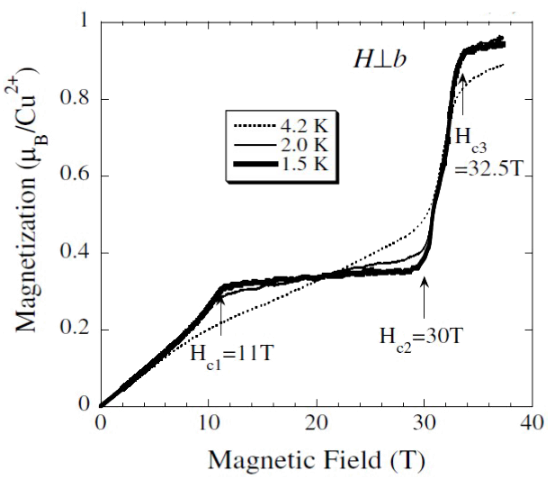
S=2



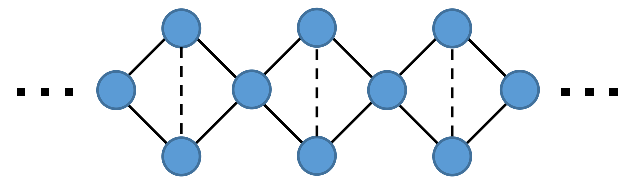
Tonegawa et al., 2011

Magnetization plateau

Region where *M* is unchanged with increasing *H* in magnetization curves



Azurite
(S=1/2 diamond chain)



Kikuchi et al., 2005

Oshikawa-Yamanaka-Affleck condition

Oshikawa-Yamanaka-Affleck, 1997

$$r(S-m) \in \mathbb{Z}$$

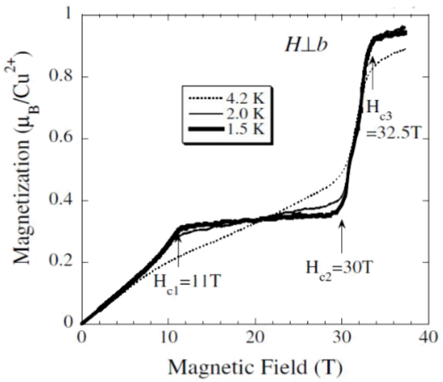
Number of sites in a unit cell

Size of spin

Magnetization per site

$$r = 3$$
$$S = 1/2$$
$$m = 1/6$$

Magnetization plateau



Magnetization plateaus are gapped states.

Q: Magnetization plateau can be considered as a SPT phase?

1. Field theory
2. Matrix product state (MPS) representation
3. Numerical calculations

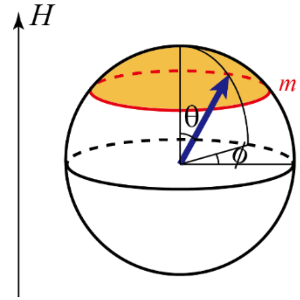
Field theory of magnetization plateau

Tanaka-Totsuka-Hu, 2009

Model $\mathcal{H} = J\sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + D\sum_j (S_j^z)^2 - H\sum_j S_j^z \quad J > 0$

Canted spin configuration

$$\boldsymbol{S}_j(\tau) = S\begin{pmatrix} (-1)^j \cos\phi_j(\tau)\sin\theta_0 \\ (-1)^j \sin\phi_j(\tau)\sin\theta_0 \\ \cos\theta_0 \end{pmatrix}$$



$$m = S\cos\theta_0 \quad \cos\theta_0 = H/(2S(D+2J))$$

$$\mathcal{S} = \mathcal{S}_{\text{kin}} + \mathcal{S}_{\text{BP}}^{\text{tot}}, \quad \mathcal{S}_{\text{kin}} = \int d\tau \mathcal{H} \xrightarrow{\text{Continuum limit}} \int dx d\tau \frac{\zeta}{2}\left\{\frac{1}{v^2}(\partial_\tau\phi)^2 + (\partial_x\phi)^2\right\}$$

$$\zeta = aJS^2\left(1 - \frac{H^2}{4S^2(D+2J)^2}\right), \quad v = Ja\sqrt{\frac{4S^2(D+2J)^2 - H^2}{2J(D+2J)}}$$

Berry phase term

$$\mathcal{S}_{\text{BP}}^{\text{tot}} = \sum_j iS(1-\cos\theta_0)\int d\tau\partial_\tau\phi_j = \sum_j i(S-m)\int d\tau\partial_\tau\phi_j$$

Berry phase term

$$\mathcal{S}_{\mathrm{BP}}^{\mathrm{tot}} = \underbrace{\sum_j (-1)^j \mathcal{S}_{\mathrm{BP},j}}_{\text{Staggered part}} + \underbrace{\sum_{j:\mathrm{odd}} 2i(S-m)\int d\tau \partial_\tau \phi_j}_{\text{Uniform part}} \qquad \mathcal{S}_{\mathrm{BP},j} = i(S-m)\int d\tau \partial_\tau \phi_j$$

Staggered part　Uniform part

Uniform part

$$\sum_{j:\mathrm{odd}} 2i(S-m)\int d\tau \partial_\tau \phi_j \to i\int dx d\tau \frac{S-m}{a}\partial_\tau \phi$$

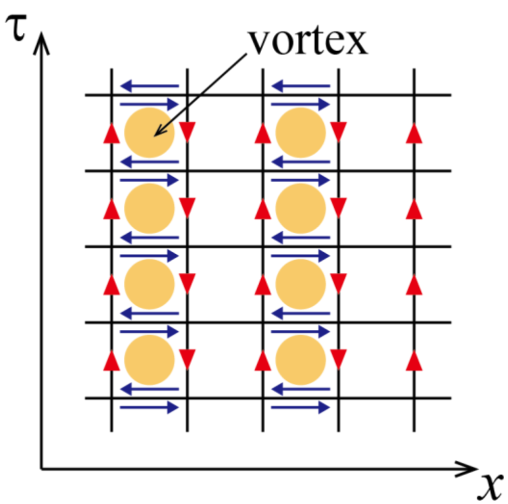
$S-m \notin \mathbb{Z}$: Gapless theory

$S-m \in \mathbb{Z}$: Magnetization plateau

Oshikawa-Yamanaka-Affleck, 1997
Tanaka-Totsuka-Hu, 2009

Staggered part

$$\begin{aligned}\mathcal{S}_{\mathrm{BP}}^{\mathrm{tot}} &= \sum_j (-1)^j (S-m)\int d\tau \partial_\tau \phi_j \\ &= i2\pi(S-m)\sum_{\substack{\mathrm{odd}\\ \mathrm{column}}} (\text{spacetime vorticity of } \phi).\end{aligned}$$



Introduction of horizontal arrows

Berry phase term

$$\mathcal{S}_{\mathrm{BP}}^{\mathrm{tot}} = i2\pi(S-m) \sum_{\substack{\text{odd}\\ \text{column}}} (\text{spacetime vorticity of } \phi)$$

$$\xrightarrow{\text{Continuum limit}} i\frac{S-m}{2}\int d\tau dx (\partial_\tau\partial_x - \partial_x\partial_\tau)\phi(\tau,x)$$

$$\boldsymbol{N}_{\mathrm{planar}}(\tau,x) \equiv \begin{pmatrix} \cos\phi(\tau,x) \\ \sin\phi(\tau,x) \\ 0 \end{pmatrix}$$

CP^1 representation $\quad N^a = \boldsymbol{z}^\dagger \sigma^a \boldsymbol{z} \qquad \boldsymbol{z} \equiv \begin{pmatrix} 1/\sqrt{2} \\ e^{i\phi(\tau,x)}/\sqrt{2} \end{pmatrix}$

$$a_\mu \equiv -i\boldsymbol{z}^\dagger \partial_\mu \boldsymbol{z} = \partial_\mu\phi/2 (\mu=\tau,x)$$

$$\mathcal{S}_{\mathrm{BP}}^{\mathrm{tot}} = i(S-m)\int d\tau dx (\partial_\tau a_x - \partial_x a_\tau)$$

cf. θ-term of O(3) nonlinear sigma model

$$\mathcal{S}_\theta = i\frac{\theta}{2\pi}\int d\tau dx (\partial_\tau a_x - \partial_x a_\tau) \longleftarrow i\frac{\theta}{4\pi}\int d\tau dx \epsilon^{abc} N_a \partial_\tau N_b \partial_x N_c$$

Effective vacuum angle $\boxed{\theta_{\mathrm{eff}} = 2\pi(S-m)}$

Groundstate wave functional

Xu-Senthil, 2013

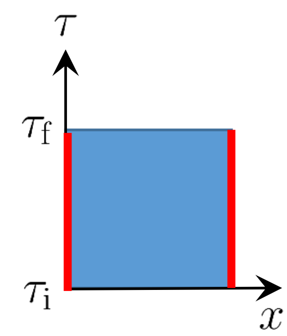
$$\mathcal{S} = \int d\tau dx \left[\frac{1}{2g}(\partial_\mu\phi)^2 + i(S-m)(\partial_\tau a_x - \partial_x a_\tau)\right]$$

CP[1] gauge fields $a_\mu \equiv \partial_\mu\phi/2(\mu = \tau, x)$

Strong coupling limit $g \to \infty$

$\Psi[\boldsymbol{N}(x)]$: probability amplitude of the configuration $\{\boldsymbol{N}(x)\}$

Path integral formalism



$$\Psi[\boldsymbol{N}(x)] \propto \int_{\boldsymbol{N}_{\rm i}}^{\boldsymbol{N}_{\rm f}=\boldsymbol{N}(x)} \mathcal{D}\boldsymbol{N}'(\tau,x) e^{-\mathcal{S}[\boldsymbol{N}'(\tau,x)]}$$

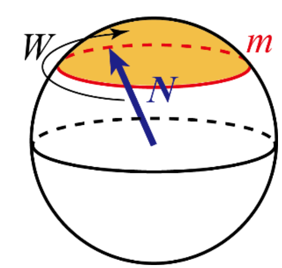
Initial and final imaginary time $\tau_{\rm i(f)}$

$$= \int_{\boldsymbol{N}_{\rm i}}^{\boldsymbol{N}(x)} \mathcal{D}\boldsymbol{N}'(\tau,x) e^{-i(S-m)\int dx(a_x(\tau_{\rm f}) - a_x(\tau_{\rm i}))}$$

$$\boxed{\Psi[\boldsymbol{N}(x)] \propto e^{-i(S-m)\pi W}}$$

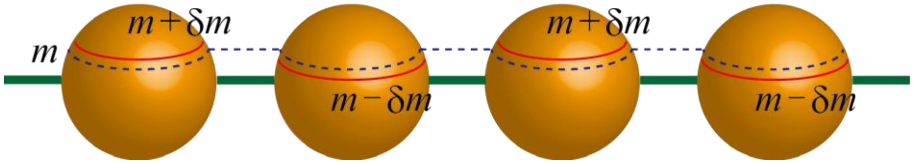
$$= \begin{cases} (-1)^W & \text{if } S-m = \text{odd} \\ 1 & \text{if } S-m = \text{even} \end{cases}$$

$$W \equiv \frac{1}{2\pi}\int dx \partial_x\phi \in \mathbb{Z}$$



Winding number of
the planar configuration

SPT breaking perturbation



Staggered field introduces z-component change δm

i.e. Modification $S-m \;\rightarrow\; S-m-\delta m$

$$\mathcal{S}^{\mathrm{tot}}_{\mathrm{BP}} = i(S-m-\delta m)\int d\tau dx(\partial_\tau a_x - \partial_x a_\tau)$$

$$\Psi[\boldsymbol{N}(x)] \propto e^{-i(S-m-\delta m)\pi W}$$

$S-m=$ even and odd are continuously connected by changing δm. → Need to check that the gap does not close. (below)

Protected by link-center inversion symmetry

Dual boson-vortex theory

$$\mathcal{S} = \int d\tau dx \Big[\frac{\zeta}{2}\Big\{\frac{1}{v^2}(\partial_\tau\phi)^2 + (\partial_x\phi)^2\Big\} + i(S-m)(\partial_\tau a_x - \partial_x a_\tau)\Big]$$

$$\mathcal{L} = \frac{1}{2g}(\partial_\mu\phi)^2 + i\pi(S-m)\rho_{\mathrm{v}}$$

$\rho_{\mathrm{v}} \equiv (\partial_\tau\partial_x - \partial_x\partial_\tau)\phi/(2\pi)$: Density of spacetime vortices

Hubbard-Stratonovich transformation

$$(\partial_\mu\phi)^2/(2g) \to (g/2)J_\mu^2 + iJ_\mu\partial_\mu\phi$$

$\phi = \phi_{\mathrm{r}} + \phi_{\mathrm{v}}$ $\quad (\partial_\tau\partial_x - \partial_x\partial_\tau)\phi_{\mathrm{r}} = 0$: Regular part

$(\partial_\tau\partial_x - \partial_x\partial_\tau)\phi_{\mathrm{v}} \neq 0$: Vortex part

Integration about ϕ_{r} → Delta function $\propto \delta(\partial_\mu J_\mu)$

$J_\mu = \epsilon_{\mu\nu}\partial_\nu\varphi/2\pi$ $\quad \varphi$: Vortex-free scalar field

$$\mathcal{L} = \frac{g}{8\pi^2}(\partial_\mu\varphi)^2 + i\pi(S - m + \varphi/\pi)\rho_{\mathrm{v}}$$

Dual boson-vortex theory

$$\mathcal{L} = \frac{g}{8\pi^2}(\partial_\mu \varphi)^2 + i\pi(S - m + \varphi/\pi)\rho_{\rm v}$$

Small fugacity expansion : Restrict the vorticity within $\rho_{\rm v} = \pm 1$

$z = e^{-\mu}$ μ : creation energy of a vortex

Lagrangian density for the vortex gas

$$\mathcal{L} = \frac{g}{2}(\partial_\mu \varphi)^2 + 2z\cos\big(\pi(S-m) + \varphi\big)$$

In a magnetization plateau,

$$\cos\big(\pi(S-m) + \varphi\big) = (-1)^{S-m}\cos\varphi$$

Final form of the vortex field theory :

$$\mathcal{L}_{\rm dual} = \frac{g}{2}(\partial_\mu \varphi)^2 + (-1)^{S-m} 2z\cos\varphi$$

sine-Gordon theory

Parity of S-m changes the phase locking point of φ

Dual boson-vortex theory

$$\mathcal{L} = \frac{g}{8\pi^2}(\partial_\mu \varphi)^2 + i\pi(S - m + \varphi/\pi)\rho_{\mathrm{v}}$$

Small fugacity expansion : Restrict the vorticity within $\rho_{\mathrm{v}} = \pm 1$

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Lagrangian density for the vortex gas

$$\mathcal{L} = \frac{g}{2}(\partial_\mu \varphi)^2 + 2z\cos\big(\pi(S - m) + \varphi\big)$$

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Final form of the vortex field theory :

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Parity of S-m changes the phase locking point of φ

If staggered field is introduced

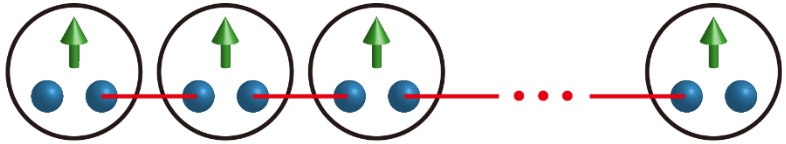
$$S - m \;\to\; S - m - \delta m$$

$$\mathcal{L} = \frac{g}{2}(\partial_\mu \varphi)^2 + 2z\cos\big(\varphi + \pi(S - m - \delta m)\big)$$

Phase locking point change continuously

MPS representation of plateaux

VBS picture for m=1/2 plateau in S=3/2



Schwinger boson representation

$$|\Psi\rangle = \prod_j P_j a_j^\dagger (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger) \bigotimes_j |0\rangle_j$$

up   down

P_j : Projection operator

$$(a_j^\dagger)^3 |0\rangle_j \to \sqrt{6} |S^z = 3/2\rangle_j$$

$$(a_j^\dagger)^2 b_j^\dagger |0\rangle_j \to \sqrt{2} |S^z = 1/2\rangle_j$$

$$a_j^\dagger (b_j^\dagger)^2 |0\rangle_j \to \sqrt{2} |S^z = -1/2\rangle_j$$

MPS representation of plateaux

Matrix product state (MPS)

$$|\Psi\rangle = \sum_{S_j^z=-3/2}^{3/2} \dots \Lambda\Gamma[S_{j-1}^z]\Lambda\Gamma[S_j^z]\Lambda\Gamma[S_{j+1}^z]\Lambda \dots \bigotimes_j |S_j^z\rangle$$

$$\Gamma[3/2] = (1+\sqrt{3})^{-1/2} \begin{pmatrix} 0 & 0 \\ -\sqrt{2}\,3^{1/4} & 0 \end{pmatrix}$$

$$\Gamma[1/2] = (1+\sqrt{3})^{-1/2} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}$$

$$\Gamma[-1/2] = (1+\sqrt{3})^{-1/2} \begin{pmatrix} 0 & \sqrt{2}\,3^{1/4} \\ 0 & 0 \end{pmatrix}$$

$$\Gamma[-3/2] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

MPS representation of plateaux

$$|\Psi\rangle = \sum_{S_j^z=-3/2}^{3/2} \ldots \Lambda\Gamma[S_{j-1}^z]\Lambda\Gamma[S_j^z]\Lambda\Gamma[S_{j+1}^z]\Lambda \ldots \bigotimes_j |S_j^z\rangle$$

Degrees of freedom of MPS

Applying a phase factor: $e^{i\theta}$

Unitary transformation: $\Lambda\Gamma \rightarrow U^\dagger(\Lambda\Gamma)U$

"Projective representation"

Link-center inversion $\mathcal{I}$ acts on MPS as $\Gamma^{\mathrm{T}} = e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^\dagger \Gamma U_{\mathcal{I}}$

$$(A_1 A_2 \cdots A_n)^{\mathrm{T}} = A_n^{\mathrm{T}} \cdots A_2^{\mathrm{T}} A_1^{\mathrm{T}}$$

$$\left.\begin{aligned} \Gamma^{\mathrm{T}} &= e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^\dagger \Gamma U_{\mathcal{I}} \\ \Gamma &= e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^{\mathrm{T}} \Gamma^{\mathrm{T}} U_{\mathcal{I}}^* \end{aligned}\right\} \ \Gamma = e^{2i\theta_{\mathcal{I}}} (U_{\mathcal{I}}^* U_{\mathcal{I}})^\dagger \Gamma U_{\mathcal{I}}^* U_{\mathcal{I}}$$

$$e^{2i\theta_{\mathcal{I}}} = 1$$

$$U_{\mathcal{I}}^* U_{\mathcal{I}} = e^{i\phi} E \rightarrow U_{\mathcal{I}} = e^{i\phi} U_{\mathcal{I}}^{\mathrm{T}} \rightarrow U_{\mathcal{I}} = \pm U_{\mathcal{I}}^{\mathrm{T}}$$

MPS representation of plateaux

Matrix product state (MPS)

$$|\Psi\rangle = \sum_{S_j^z=-3/2}^{3/2} \ldots \Lambda\Gamma[S_{j-1}^z]\Lambda\Gamma[S_j^z]\Lambda\Gamma[S_{j+1}^z]\Lambda \ldots \bigotimes_j |S_j^z\rangle$$

$$\Gamma[3/2] = (1+\sqrt{3})^{-1/2}\begin{pmatrix} 0 & 0 \\ -\sqrt{2}\,3^{1/4} & 0 \end{pmatrix} \qquad \Gamma[1/2] = (1+\sqrt{3})^{-1/2}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}$$

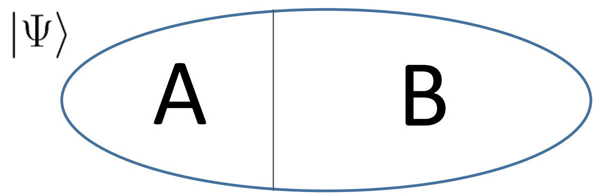
$$\Gamma[-1/2] = (1+\sqrt{3})^{-1/2}\begin{pmatrix} 0 & \sqrt{2}\,3^{1/4} \\ 0 & 0 \end{pmatrix} \qquad \Gamma[-3/2] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$$

$$\Gamma^{\mathrm{T}} = e^{i\theta_{\mathcal{I}}} U_{\mathcal{I}}^{\dagger} \Gamma U_{\mathcal{I}}$$

We can find $U_{\mathcal{I}} = i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\boxed{U_{\mathcal{I}} = -U_{\mathcal{I}}^{\mathrm{T}} \text{ : Nontrivial}}$

Entanglement spectrum



Schmidt decomposition $|\Psi\rangle = \sum_{\alpha} \lambda_\alpha |\Psi_\mathrm{A}\rangle_\alpha \otimes |\Psi_\mathrm{B}\rangle_\alpha$

$\rho_\mathrm{A} \equiv \mathrm{Tr}_\mathrm{B} |\Psi\rangle\langle\Psi| = \sum_{\alpha} \lambda_\alpha^2 |\Psi_\mathrm{A}\rangle_{\alpha\,\alpha}\langle\Psi_\mathrm{A}|$: Density matrix

Entanglement spectrum (ES)

$\{-\ln(\lambda_\alpha^2)\} \quad (\alpha = 1, \dots, \chi)$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \\ 0 & \lambda_2 & 0 & \\ 0 & 0 & \lambda_3 & \\ & & & \ddots \end{pmatrix}$$

$U_\mathcal{I}$: block diagonal about singular values (subspace index k)

$$U_\mathcal{I} = -U_\mathcal{I}^\mathrm{T}$$

dimension of each block

$$\det(U_{\mathcal{I},k}) = \det(U_{\mathcal{I},k}^\mathrm{T}) = \det(-U_{\mathcal{I},k}) = (-1)^{d_k} \det(U_{\mathcal{I},k})$$

d_k should be even $\rightarrow$ ES is two-fold degenerate

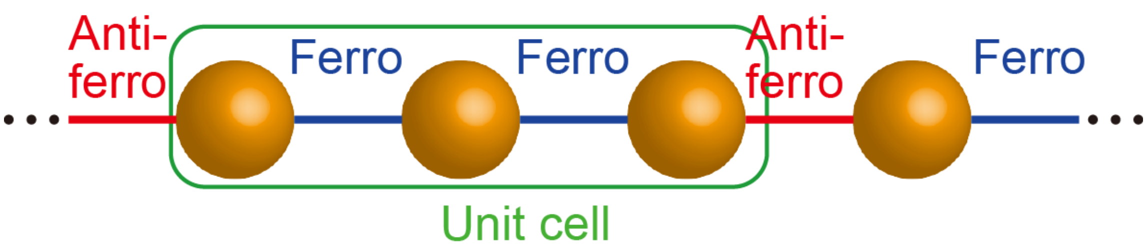
Numerical calculations in FFAF chains

Studying the model $\mathcal{H} = J\sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + D\sum_j (S_j^z)^2 - H\sum_j S_j^z$ is preferable.

However, it is difficult to investigate plateaux in this model (very small region).

It is easier to find magnetization plateaux in ferro-ferro-antiferromagnetic (FFAF) chains.

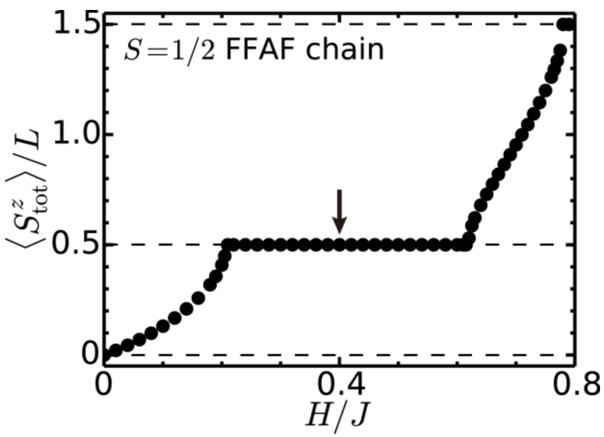
Hida, 1994

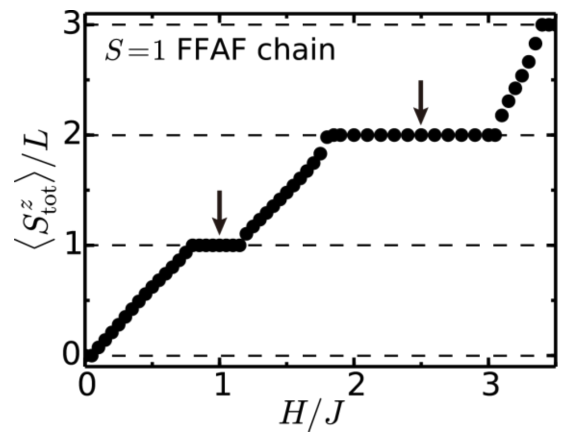


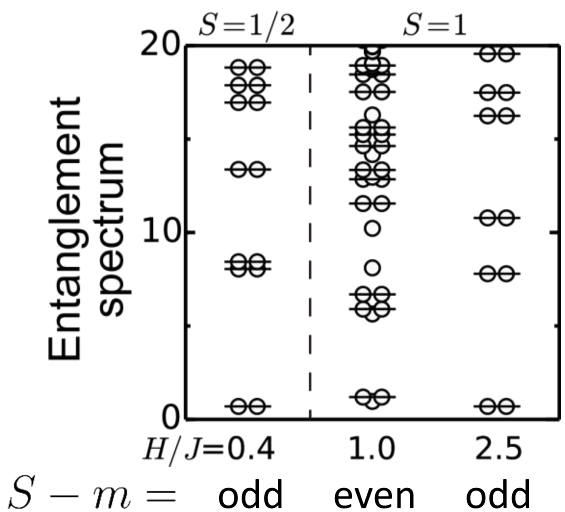
Candidate material : $Cu_3(P_2O_6OH)_2$

Numerical calculations in FFAF chains

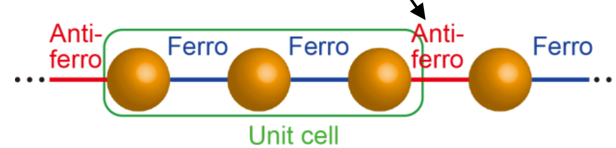
- Infinite-time evolving block decimation (iTEBD).
- Magnetization curves and entanglement spectra.







Bipartition at the AF bond



Conclusion

Magnetization plateau states in 1D antiferromagnets are in an SPT phase protected by link-inversion symmetry if S-m = odd integer.

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1. Field theories
   – Nonlinear sigma model with topological term
   – Wave functional by path integral
2. Matrix product state (MPS) representation
3. Numerical calculations
   – Structure of entanglement spectrum